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AN UPPER BOUND FOR d-DIMENSIONAL DIFFERENCE SETS

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Let s_d be the maximal positive number for which the inequality $|\mathbb{A}-\mathbb{A}|\geq s_d|\mathbb{A}|-t_d$ holds for every finite set $\mathbb{A}$ of affine dimension $\dim\mathbb{A}=d$. What can one say about s_d ? The exact value of s_d is known only for $d=1,2$ and 3. It is shown that $s_d\leq 2d-2+\frac{1}{d-1}$, for every $d\geq 2$. This disproves a conjecture of Ruzsa. Some further related questions are posed and discussed.

1. Introduction

Let $\mathbb{A}$ be a finite set of $\mathbb{R}^d$. Let $\mathbb{A}-\mathbb{A}=\{x-y : x,y\in\mathbb{A}\}$ denote the difference set of $\mathbb{A}$. The need for lower estimates for $|\mathbb{A}-\mathbb{A}|$ in terms of $|\mathbb{A}|$ has been raised by Uhrin in [6]. In general, nothing more than the obvious

$$|\mathbb{A}-\mathbb{A}|\geq 2|\mathbb{A}|-1 \tag{1}$$

can be asserted; this lower bound, which holds with equality if $\mathbb{A}$ is an arithmetic progression, was used in [6] and [7] to prove results sharpening the classical theorem of Minkowski-Blichfeld in geometry of numbers.

Therefore, a natural problem consists of obtaining non-trivial estimates for $|\mathbb{A}-\mathbb{A}|$, taking into consideration additional information about the structure of $\mathbb{A}$. We introduce the following definition. Let $s(d)$ be the maximal positive number for which there is $t(d)$ such that the inequality

$$|\mathbb{A}-\mathbb{A}|\geq s(d)|\mathbb{A}|-t(d) \tag{2}$$

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holds for every finite set $\mathbb{A}$ of affine dimension $\dim \mathbb{A} = d$. It is known [2] that $|\mathbb{A} - \mathbb{A}| \geq (d+1)|\mathbb{A}| - \frac{1}{2}d(d+1)$, if $\dim \mathbb{A} = d$. Therefore, one has for every d

$$s(d) \geq d+1. \tag{3}$$

In particular, if $d=1$, then $|\mathbb{A}-\mathbb{A}| \geq 2|\mathbb{A}|-1$ and if $d=2$, then $|\mathbb{A}-\mathbb{A}| \geq 3|\mathbb{A}|-3$. These two inequalities cannot be strengthened (see examples 1 and 2 in [4]). Consequently we get

$$s(1) = 2 \quad \text{and} \quad s(2) = 3. \tag{4}$$

However, the lower bound (3) is not exact for $d=3$. We showed in [4] that for finite three-dimensional sets the "correct" lower bound is

$$|\mathbb{A} - \mathbb{A}| \geq 4.5|\mathbb{A}| - 9\ . \tag{5}$$

This inequality, which was conjectured earlier by Freiman, Heppes and Uhrin in [2] and by Ruzsa in [3], is tight (see example 3 in [4]). Therefore, we get

$$s(3) = 4.5\ . \tag{6}$$

So far $s(1), s(2)$ and $s(3)$ are the only known values of $s(d)$. What can one say about $s(d)$ for $d \geq 4$? The lower bound (3), obtained by Freiman, Heppes and Uhrin, is probably not exact. Ruzsa conjectured in [3] that $s(d) = 2d - 2 + \frac{2}{d}$, for every $d \geq 4$.

The aim of this paper is to prove the following upper bound for $s(d)$:

Theorem 1.1. *For every integer $d \geq 2$ one has*

$$s(d) \leq 2d - 2 + \frac{1}{d-1}\ . \tag{7}$$

This readily disproves Ruzsa's conjecture. Moreover, in view of (4) and (6), it seems that the equality $s(d) = 2d - 2 + \frac{1}{d-1}$ is true for every $d \geq 2$. Thus, we suggest the following

Conjecture 1.2. For every finite set $\mathbb{A}$ of affine dimension $\dim \mathbb{A} = d \geq 2$, one has

$$|\mathbb{A} - \mathbb{A}| \geq (2d - 2 + \frac{1}{d-1})|\mathbb{A}| - (2d^2 - 4d + 3). \tag{8}$$

Of course, in view of Theorem 1.1, if inequality (8) is true, then is best possible (see also equality (13) below).

In this paper we shall use the following notation. Let A and B be two subsets of $\mathbb{R}^m$. As usual their *sum* is defined by $A + B = \{a + b, a \in A, b \in B\}$, $-A$ means the set $\{-x : x \in A\}$, $A + A$ is called the *sum set* of A and

$A-A=A+(-A)$ is the *difference set* of A. If X is a subset of $\mathbb{R}^m$ and $a\in\mathbb{R}^m$, we write $a+X$ for the set $\{a+x : x\in X\}$. If X is a finite set, we denote by $|X|$ its *cardinality*. The *affine dimension* $\dim A$ of a set $A\subseteq\mathbb{R}^m$ is defined as the dimension of the smallest hyperplane containing A. We denote by $e_0=(0,0,\ldots,0)$, $e_1=(1,0,\ldots,0),\ldots,e_m=(0,0,\ldots,1)$ the $(m+1)$ vertices of the standard m-simplex $\mathbb{S}_m\subseteq\mathbb{R}^m$. A vector will be written in the form $(x_1,\ldots,x_m)$, where x_i, $1\le i\le m$, are its coordinates with respect to the canonical basis $\{e_1,\ldots,e_m\}$.

2. The upper bound for $s(d)$

We begin with two simple observations.

(i) Let $\mathbb{S}_m=\{e_0,e_1,\ldots,e_m\}$ be a $m-$dimensional simplex in $\mathbb{R}^m$. Clearly $|\mathbb{S}_m|=m+1$ and $\mathbb{S}_m$ is a "generic" set: there is no nontrivial coincidence between sums and differences. Therefore, we have

$$|\mathbb{S}_m-\mathbb{S}_m|=|\mathbb{S}_m|^2-|\mathbb{S}_m|+1=m^2+m+1, \tag{9}$$

$$|\mathbb{S}_m+\mathbb{S}_m|=(m+1)|\mathbb{S}_m|-\frac{1}{2}m(m+1)=\frac{1}{2}(m+1)(m+2). \tag{10}$$

(ii) We estimate $|X\pm X|$ for a set which consists of the union of k parallel sets of dimension $m-1$. Let B be a finite set included in the hyperplane $(x_m=0)$ and define $X\subseteq\mathbb{R}^m$ by $X=X_k=B\cup(B+e_m)\cup\ldots\cup(B+(k-1)e_m)$. We can write

$$X-X=(B-B)+\{te_m : -(k-1)\le t\le k-1, t\in\mathbb{Z}\},$$
$$X+X=(B+B)+\{te_m : 0\le t\le 2k-2, t\in\mathbb{Z}\},$$

and hence $|X\pm X|=(2k-1)|B\pm B|$.

Remark. Using **(ii)** with $B=\mathbb{S}_{m-1}$, we obtain a m-dimensional set

$$X=\mathbb{S}_{m-1}\cup(\mathbb{S}_{m-1}+e_m)\cup\ldots\cup(\mathbb{S}_{m-1}+(k-1)e_m),$$

which consists of m parallel arithmetic progressions with the same common difference e_m, having $|X|=mk$. Using (9) and (10) for $\mathbb{S}_{m-1}$, we get

$$|X+X|=(2k-1)|\mathbb{S}_{m-1}+\mathbb{S}_{m-1}|=(m+1)|X|-\frac{1}{2}m(m+1), \tag{11}$$

$$|X-X|=(2k-1)|\mathbb{S}_{m-1}-\mathbb{S}_{m-1}|=\left(2m-2+\frac{2}{m}\right)|X|-(m^2-m+1). \tag{12}$$

Inequality (11) shows that Freiman's lower bound for the sumset ([1], pp.24) $|\mathbb{C}+\mathbb{C}|\ge(m+1)|\mathbb{C}|-\frac{m(m+1)}{2}$, with $\dim\mathbb{C}=m$, is best possible and we obtained in [5] a complete description of m-dimensional sets having the smallest cardinality of the sumset. Moreover, inequality (12), with $m=d$, gives only a first estimate for $s(d)$, namely $s(d)\leq 2d-2+\frac{2}{d}$. Nevertheless, we will improve it by using a symmetric set M lying on two parallel hyperplanes. ∎

Proof. Let $d\geq 2$ be an integer. We construct for every positive integer k, a set $M=M_k\subseteq\mathbb{R}^d$ of affine dimension d, cardinality $|M|=2k(d-1)$ and

$$|M-M|=(2d-2+\frac{1}{d-1})|M|-(2d^2-4d+3). \tag{13}$$

Sending k to infinity and in view of the definition of $s(d)$, we obtain (7).

Let $M=Y\cup(e_d-Y)$ where $Y=\mathbb{S}_{d-2}\cup(\mathbb{S}_{d-2}+e_{d-1})\cup\ldots\cup(\mathbb{S}_{d-2}+(k-1)e_{d-1})$. Note that M lies on two parallel hyperplanes $(x_d=0)$ and $(x_d=1)$, has the cardinality $|M|=2|Y|=2k|\mathbb{S}_{d-2}|=2k(d-1)$ and $\dim M=\dim Y+1=d$.

We have $M-M=(Y-Y)\cup((Y+Y)-e_d)\cup(e_d-(Y+Y))$ and these three components of $M-M$ lie on different hyperplanes. Therefore, one has $|M-M|=|Y-Y|+2|Y+Y|$. In view of **(ii)** with $B=\mathbb{S}_{d-2}$, we get $|Y\pm Y|=(2k-1)|\mathbb{S}_{d-2}\pm\mathbb{S}_{d-2}|$. Using (9) and (10) with $m=d-2$ we get

$$\begin{aligned}
|M-M| &= |Y-Y|+2|Y+Y|\\
&= (2k-1)|\mathbb{S}_{d-2}-\mathbb{S}_{d-2}|+2(2k-1)|\mathbb{S}_{d-2}+\mathbb{S}_{d-2}|\\
&= (2k-1)(d^2-3d+3)+2(2k-1)\frac{d^2-d}{2}\\
&= (2k-1)(2d^2-4d+3)\\
&= (2d-2+\frac{1}{d-1})|M|-(2d^2-4d+3).
\end{aligned}$$

The theorem is proved. A final observation: M consists of $2d-2$ parallel arithmetic progressions with the same common difference. Indeed, we can write $M=T\cup(T+b)\cup\ldots\cup(T+(k-1)b)$, where $T=\mathbb{S}_{d-2}\cup(a-\mathbb{S}_{d-2})$, a does not lie in the hyperplane of $\mathbb{S}_{d-2}$ and b does not lie in the hyperplane of T. ∎

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